

Hypergroups with Unique α -Means

Ahmadreza Azimifard

Abstract:

Let K be a commutative hypergroup and $\alpha \in \widehat{K}$. We show that K is α -amenable with the unique α -mean m_α if and only if $m_\alpha \in L^1(K) \cap L^2(K)$ and α is isolated in \widehat{K} . In contrast to the case of amenable noncompact locally compact groups, examples of polynomial hypergroups with unique α -means ($\alpha \neq 1$) are given. Further examples emphasize that the α -amenability of hypergroups depends heavily on the asymptotic behavior of Haar measures and characters.¹

Résumé:

Soit K un hypergroupe commutatif et $\alpha \in \widehat{K}$. Nous montrons que K est α -moyennable avec unicité de l' α -moyenne m_α si et seulement si $m_\alpha \in L^1(K) \cap L^2(K)$ et α est isolé dans \widehat{K} . Contrairement au cas des groupes moyennables localement compacts mais non compacts, des exemples d'hyper-groupes polynomiaux avec unicité des α -moyennes ($\alpha \neq 1$) sont donnés. Nous montrons à l'aide d'autres exemples que l' α -moyennabilité des hypergroupes dépend fortement de leurs mesures de Haar ainsi que du comportement du caractères.

Keywords. Hypergroups: orthogonal polynomial, of Nevaei classes.
 α -Amenable Hypergroups.

AMS Subject Classification 2000: primary 43A62, 43A07, secondary 46H20.

¹ Parts of the paper are taken from the author's Ph.D. thesis at the Technical University of Munich.

1 Introduction

Recently the notion of α -amenable hypergroups was introduced and studied in [8]. Let K be a commutative locally compact hypergroup and let $L^1(K)$ denote the hypergroup algebra. Assume that $\alpha \in \widehat{K}$ and denote by $I(\alpha)$ the maximal ideal in $L^1(K)$ generated by α . As shown in [8], K is α -amenable if and only if either $I(\alpha)$ has a b.a.i. (bounded approximate identity) or K satisfies the modified Reiter's condition of P_1 -type in α . Commutative hypergroups are always 1-amenable [14], whereas a large class of non α -amenable hypergroups, $\alpha \neq 1$, are given in [1, 8]. It is worth to notify that $1 \in \text{supp } \pi_K$ does not hold in general, where $\text{supp } \pi_K$ denotes the support of the Plancherel measure on \widehat{K} [10, 14].

As in the case of locally compact groups [13], if K is a noncompact locally compact amenable hypergroup, then the cardinality of (1-)means is 2^{2^d} , where d is the smallest cardinality of a cover of K by compact sets [14]. However, it is well known that K has a unique (1-)mean if and only if K is compact [13, 14]. Hence, $\text{supp } \pi_K = \widehat{K}$ and K is α -amenable for every $\alpha \in \widehat{K}$ [2, 8].

For a α -amenable hypergroup K with a unique α -mean, one can pose the natural question of whether K is compact or K is β -amenable when $\alpha \neq \beta \in \widehat{K}$. Theorem 3.1 answers this question completely. In addition, examples of polynomial hypergroups show that the α -amenability of hypergroups depends on the asymptotic behavior of the Haar measures and characters. Furthermore, the α -amenability of K with a unique α -mean ($\alpha \neq 1$), even in every $\alpha \in \widehat{K} \setminus \{1\}$, does not imply the compactness of K ; see Section 4.

Different axioms for hypergroups are given in [7, 10, 15]. However, in this paper we refer to Jewett's axioms in [10].

2 Preliminaries

Let (K, ω, \sim) be a locally compact hypergroup, where $\omega : K \times K \rightarrow M^1(K)$ defined by $(x, y) \mapsto \omega(x, y)$, and $\sim : K \rightarrow K$ defined by $x \mapsto \tilde{x}$, denote the convolution and involution on K , where $M^1(K)$ stands for the set of all probability measures on K . The hypergroup K is called commutative if $\omega(x, y) = \omega(y, x)$ for every $x, y \in K$.

Throughout this paper K is a commutative hypergroup. Let $C_c(K)$ be the spaces of all continuous functions on K with compact support. The translation of $f \in C_c(K)$ at the point $x \in K$, $T_x f$, is defined by

$$T_x f(y) := \int_K f(t) d\omega(x, y)(t), \text{ for every } y \in K.$$

Being K commutative ensures the existence of a Haar measure m on K which is unique up to

a multiplicative constant [15]. Let $(L^p(K), \|\cdot\|_p)$ ($p = 1, 2$) denote the usual Banach space of Borel measurable functions on K [10, 6.2]. For $f, g \in L^1(K)$ we may define the convolution and involution by $f * g(x) := \int_K f(y)T_{\tilde{y}}g(x)dm(y)$ (m -a.e. on K) and $f^*(x) = \overline{f(\tilde{x})}$, respectively, that $(L^1(K), \|\cdot\|_1)$ becomes a Banach $*$ -algebra. If K is discrete, then $L^1(K)$ has an identity element. Otherwise $L^1(K)$ has a b.a.i., i.e. there exists a net $\{e_i\}_i$ of functions in $L^1(K)$ with $\|e_i\|_1 \leq M$, for some $M > 0$, such that $\|f * e_i - f\|_1 \rightarrow 0$ as $i \rightarrow \infty$ [2]. The set of all multiplicative linear functionals on $L^1(K)$, i.e. the maximal ideal space of $L^1(K)$ [3], can be identified with

$$\mathfrak{X}^b(K) := \left\{ \alpha \in C^b(K) : \alpha \neq 0, \omega(x, y)(\alpha) = \alpha(x)\alpha(y), \forall x, y \in K \right\}$$

via $\varphi_\alpha(f) := \int_K f(x)\overline{\alpha(x)}dm(x)$, for every $f \in L^1(K)$. $\mathfrak{X}^b(K)$ is a locally compact Hausdorff space with the compact-open topology [2]. $\mathfrak{X}^b(K)$ and its subset

$$\widehat{K} := \{ \alpha \in \mathfrak{X}^b(K) : \alpha(\tilde{x}) = \overline{\alpha(x)}, \forall x \in K \}$$

are considered as the character spaces of K . The maximal ideal in $L^1(K)$ generated by the character α is $I(\alpha) := \{f \in L^1(K) : \varphi_\alpha(f) = 0\}$. The Fourier transform of $f \in L^1(K)$, $\widehat{f} \in C_0(\widehat{K})$, is $\widehat{f}(\alpha) := \varphi_\alpha(f)$ for every $\alpha \in \widehat{K}$. There exists a unique (up to a multiplicative constant) regular positive Borel measure π_K on \widehat{K} with $\text{supp } \pi_K = \mathcal{S}$ such that $\int_K |f(x)|^2 dm(x) = \int_{\mathcal{S}} |\widehat{f}(\alpha)|^2 d\pi_K(\alpha)$ for all $f \in L^1(K) \cap L^2(K)$ [2]. The extension of the Fourier transform defined on $L^1(K) \cap L^2(K)$ to all of $L^2(K)$ onto $L^2(\widehat{K})$ is the Plancherel transform which is an isometric isomorphism. Observe that \mathcal{S} is a nonvoid closed subset of \widehat{K} , and the constant function 1 is in general not contained in \mathcal{S} [10, 9.5].

The inverse Fourier transform for $\varphi \in L^1(\widehat{K})$ is given by $\check{\varphi}(x) = \int_{\mathcal{S}} \varphi(\alpha)\alpha(x)d\pi_K(\alpha)$ for every $x \in K$. Then $\check{\varphi} \in C_0(K)$ and if $\check{\varphi} \in L^1(K)$ then $\widehat{\check{\varphi}} = \varphi$ [2].

Let $L^1(K)^*$ and $L^1(K)^{**}$ denote the dual and the bidual spaces of $L^1(K)$ respectively. As usual, $L^1(K)^*$ can be identified with the space $L^\infty(K)$ of essentially bounded Borel measurable complex-valued functions on K . We may define the Arens product on $L^1(K)^{**}$ as follows:

$$\langle m \cdot m', f \rangle = \langle m, m' \cdot f \rangle$$

in which $\langle m' \cdot f, g \rangle = \langle m', f \cdot g \rangle$ and $\langle f \cdot g, h \rangle = \langle f, g * h \rangle$ for all $m, m' \in L^1(K)^{**}$, $f \in L^\infty(K)$ and $g, h \in L^1(K)$. $L^1(K)^{**}$ with the Arens product is a noncommutative Banach algebra in general [3, 5]. From the definitions of the Arens product and the convolution we may have $g \cdot f = g^* * f$ and $m \cdot (f \cdot g) = (m \cdot f) \cdot g$.

Definition 2.1. Let K be a commutative hypergroup and $\alpha \in \widehat{K}$. K is called α -amenable if there exists a bounded linear functional m_α on $L^\infty(K)$ with the following properties:

$$(i) \ m_\alpha(\alpha) = 1,$$

$$(ii) \ m_\alpha(\delta_x * f) = \alpha(x)m_\alpha(f), \quad \text{for every } f \in L^\infty(K) \text{ and } x \in K.$$

For example, if K is compact or $L^1(K)$ is amenable, then K is α -amenable, for every $\alpha \in \widehat{K}$ [8, 14].

3 Main Theorem

Theorem 3.1. Let K be a hypergroup and $\alpha \in \widehat{K}$. If K is α -amenable with the unique α -mean m_α , then

$$(i) \ m_\alpha \text{ and } \alpha \text{ belong to } L^1(K) \cap L^2(K) \text{ and } \alpha \in \mathcal{S} \text{ is isolated. Further, } m_\alpha^2 = m_\alpha.$$

$$(ii) \ m_\alpha = \pi(\alpha) / \|\alpha\|_2^2, \text{ where } \pi : L^1(K) \rightarrow L^1(K)^{**} \text{ is the canonical embedding.}$$

$$(iii) \text{ If } \alpha \text{ is positive, then } \alpha = 1, \text{ hence } K \text{ is compact.}$$

Proof. Since K is α -amenable with the unique α -mean m_α , $m_\alpha(\alpha) = 1$, and $f \cdot g = g \cdot f$, we have

$$\begin{aligned} \langle m_\alpha, f \cdot g \rangle &= \langle m_\alpha, g^* * f \rangle \\ &= \langle m_\alpha, \int_K (\delta_x * f) g^*(x) dm(x) \rangle \\ &= \int_K \langle m_\alpha, \delta_x * f \rangle g^*(x) dm(x) \\ &= \widehat{g^*}(\alpha) \langle m_\alpha, f \rangle, \end{aligned}$$

for every $f \in L^\infty(K)$ and $g \in L^1(K)$. Moreover, if $n \in L^1(K)^{**}$ and $h \in L^1(K)$, then

$$\langle m_\alpha \cdot n, f \cdot g \rangle = \langle m_\alpha, n \cdot (f \cdot g) \rangle = \langle m_\alpha, (n \cdot f) \cdot g \rangle = \widehat{g^*}(\alpha) \langle m_\alpha, n \cdot f \rangle = \widehat{g^*}(\alpha) \langle m_\alpha \cdot n, f \rangle.$$

Since the α -mean m_α is unique and the associated functional to α on $L^1(K)^{**}$ is multiplicative [5], $m_\alpha \cdot n = \lambda_n \cdot m_\alpha$, where $\lambda_n = \langle n, \alpha \rangle$. Let (n_i) be a net in $L^1(K)^{**}$ converging to n in the w^* -topology. Then the convergence $\lambda_{n_i} \rightarrow \lambda_n$, as $i \rightarrow \infty$, implies that the mapping $n \rightarrow m_\alpha \cdot n$ is w^* - w^* continuous on $L^1(K)^{**}$, hence m_α is in $L^1(K)$, the topological centre of $L^1(K)^{**}$ [11]. In that $\widehat{m_\alpha}(\alpha) = 1$, $g \cdot m_\alpha = \widehat{g^*}(\alpha) m_\alpha$ for every $g \in L^1(K)$, and the Arens product is continuous in the first variable, then $m_\alpha^2 = m_\alpha$.

Let $\beta \in \widehat{K}$. The equality $\beta(x)m_\alpha(\beta) = m_\alpha(T_x\beta) = \alpha(x)m_\alpha(\beta)$, for all $x \in K$, implies that $m_\alpha(\beta) = \delta_\alpha(\beta)$. Since $\widehat{m_\alpha} \in C_0(\widehat{K})$, α is isolated in \widehat{K} and $\widehat{m_\alpha} \in L^1(\widehat{K})$. The inverse of Fourier

theorem yields $m_\alpha = \widehat{m_\alpha}^\vee$, hence $\alpha \in \mathcal{S}$. Moreover, since the Plancherel transform is an isometric isomorphism of $L^2(K)$ onto $L^2(\widehat{K})$ and $\widehat{m_\alpha}(\beta) = \delta_\beta(\alpha)$, $m_\alpha \in L^2(K)$.

(ii) Plainly $\delta_x \cdot m_\alpha = \alpha(x)m_\alpha$, for every $x \in K$, so it follows from part (i) that $\alpha \in L^1(K) \cap L^2(K)$. Let $n_\alpha = \pi(\alpha)/\|\alpha\|_2^2$. We shall prove $m_\alpha = n_\alpha$. Apparently $\langle n_\alpha, \alpha \rangle = 1$, and for every $x \in K$ and $f \in L^\infty(K)$ we have $\langle n_\alpha, T_x f \rangle = \alpha(x)\langle n_\alpha, f \rangle$, hence n_α is a α -mean on $L^\infty(K)$. Since $m_\alpha \in L^1(K)^{**}$, there exists $(m_i)_i$ a net of functions in $L^1(K)$ such that $\pi(m_i) \xrightarrow{w^*} m_\alpha$, Goldstein's theorem [6]. Moreover, $m_j \cdot \alpha = \alpha \cdot m_j = \widehat{m_j}(\alpha)\alpha$ and $m_\alpha(\alpha) = 1$, so taking the w^* -limit yields $m_\alpha \cdot \pi(\alpha) = \pi(\alpha)$. Therefore, for every $f \in L^\infty(K)$ and $x \in K$ we have

$$\|\alpha\|_2^2 \langle n_\alpha, f \rangle = \langle \pi(\alpha), f \rangle = \langle m_\alpha \cdot \pi(\alpha), f \rangle = \langle m_\alpha, \pi(\alpha) \cdot f \rangle = \langle m_\alpha, \alpha \cdot f \rangle = \|\alpha\|_2^2 \langle m_\alpha, f \rangle,$$

hence $m_\alpha = n_\alpha$.

(iii) By (i) since $\alpha \in L^1(K) \cap L^2(K)$, we have

$$\alpha(x) \int_K \alpha(y) dm(y) = \int_K T_x \alpha(y) dm(y) = \int_K \alpha(y) dm(y),$$

which implies that $\alpha(x) = 1$ for every $x \in K$, hence K is compact [10]. □

Corollary 3.2. Let K be a α -amenable hypergroup with a unique α -mean in all $\alpha \in \widehat{K} \setminus \{1\}$. Then $1 \in \mathcal{S}$.

Remark 3.3. We observe that part (iii) of Theorem 3.1 can also be derived from part (i) and [16, Theorem 2.1].

4 Examples

(I) **Symmetric hypergroup [17]:** For each $n \in \mathbb{N}$, let $b_n \in]0, 1]$, $c_0 = 1$, and define numbers c_n inductively by $c_n = \frac{1}{b_n}(c_0 + c_1 + \dots + c_{n-1})$. A symmetric hypergroup structure on \mathbb{N}_0 is defined by $\varepsilon_n * \varepsilon_m = \varepsilon_m * \varepsilon_n = \varepsilon_n$ if $0 \leq m < n$ and

$$\varepsilon_n * \varepsilon_n = \frac{c_0}{c_n} \varepsilon_0 + \frac{c_1}{c_n} \varepsilon_1 + \dots + \frac{c_{n-1}}{c_n} \varepsilon_{n-1} + (1 - b_n) \varepsilon_n.$$

\mathbb{N}_0 with the above convolution and an involution defined by the identity map is a commutative hypergroup with $\mathfrak{X}^b(\mathbb{N}_0) = \mathcal{S}$. Every nontrivial character α in $\widehat{\mathbb{N}_0}$ has a finite support, so $\alpha \in \ell^1(\mathbb{N}_0) \cap \ell^2(\mathbb{N}_0)$. Consequently by Theorem 3.1 we see that \mathbb{N}_0 is α -amenable with a unique α -mean if and only if $\alpha \neq 1$.

(II) Let $\{p_n\}_{n \in \mathbb{N}_0}$ be a set of polynomials defined by a recursion relation

$$p_1(x)p_n(x) = a_np_{n+1}(x) + b_np_n(x) + c_np_{n-1}(x) \quad (1)$$

for $n \in \mathbb{N}$ and $p_0(x) = 1$, $p_1(x) = \frac{1}{a_0}(x - b_0)$, where $a_n > 0$, $b_n \in \mathbb{R}$ for all $n \in \mathbb{N}_0$ and $c_n > 0$ for $n \in \mathbb{N}$. There exists a probability measure $\pi \in M^1(\mathbb{R})$ such that $\int_{\mathbb{R}} p_n(x)p_m(x)d\pi(x) = \delta_{n,m}\mu_m$ ($\mu_m > 0$) [4]. Assume that $p_n(1) \neq 0$, so after renorming, for $n \in \mathbb{N}_0$ we have $p_n(1) = 1$. The relation (1) implies that $a_n + b_n + c_n = 1$ and $a_0 + b_0 = 1$. The polynomial set $\{p_n\}_{n \in \mathbb{N}_0}$ induces a hypergroup structure on \mathbb{N}_0 [2], which is known as a polynomial hypergroup.

(i) **Hypergroups of compact type [9]:** If in the recursion formula (1) $a_n, c_n \rightarrow 0$ and $b_n \rightarrow 1$ as $n \rightarrow \infty$, then the induced hypergroup \mathbb{N}_0 is called to be of compact type. In this case, $\mathcal{S} = \widehat{\mathbb{N}_0} = \mathfrak{X}^b(\mathbb{N}_0)$, 1 is the only accumulation point of $\widehat{\mathbb{N}_0}$ and nontrivial characters of \mathbb{N}_0 belong to $\ell^1(\mathbb{N}_0) \cap \ell^2(\mathbb{N}_0)$. By Theorem 3.1 we see that \mathbb{N}_0 is α -amenable with a unique α -mean if and only if $\alpha \neq 1$. For instance, the little q -Legendre polynomial hypergroup is of compact type.

(ii) **Hypergroups of Nevai Classes:** Let $\{p_n\}_{n \in \mathbb{N}_0}$ define a hypergroup structure on \mathbb{N}_0 with the relations (1). Consider the orthonormal polynomials $q_n(x) := \sqrt{h(n)}p_n(x)$, which by the recursion (1) satisfy the following recursion formula

$$xq_n(x) = \lambda_{n+1}q_{n+1}(x) + \beta_nq_n(x) + \lambda_nq_{n-1}(x), \quad \forall n \in \mathbb{N}_0,$$

where $q_0(x) = 1$, $\lambda_n = a_0\sqrt{c_n a_{n-1}}$ for $n \geq 2$, $\lambda_1 = a_0\sqrt{c_1}$, $\lambda_0 = 0$, and $\beta_n = a_0b_n + b_0$ for $n \geq 1$, with $\beta_0 = b_0$. The polynomial set $(q_n)_{n \in \mathbb{N}_0}$ is of the Nevai class $M(0, 1)$ if $\lim_{n \rightarrow \infty} \lambda_n = \frac{1}{2}$ and $\lim_{n \rightarrow \infty} \beta_n = 0$. It has a bounded variation, $(q_n)_{n \in \mathbb{N}_0} \in BV$, if

$$\sum_{n=1}^{\infty} (|\lambda_{n+1} - \lambda_n| + |\beta_{n+1} - \beta_n|) < \infty.$$

Theorem 4.1. Let $(q_n)_{n \in \mathbb{N}_0} \in BV \cap M(0, 1)$ and $\alpha_x \in \widehat{\mathbb{N}_0}$, where $\alpha_x(n) := q_n(x)$ for $n \in \mathbb{N}_0$. Then the followings hold:

- (i) $\mathcal{S} \cong [-1, 1] \cup A$, where A is a nonempty countable set and $[-1, 1] \cap A = \emptyset$.
- (ii) If $x \in A$, then \mathbb{N}_0 is α_x -amenable with a unique α_x -mean.
- (iii) If $h(n)$ is unbounded, then \mathbb{N}_0 is not α_x -amenable for $x \in (-1, 1)$.
- (iv) If $h(n)$ is bounded, then \mathbb{N}_0 is α_x -amenable for $x \in (-1, 1)$.

Proof. (i) It is shown in [12, Theorem 7].

(ii) Let \mathcal{S} be as in part (i). If $A \cap]1, \infty[\neq \emptyset$, then $x_1 := \sup A \in A$ corresponds to a positive character of \mathbb{N}_0 , [4, Theorem 5.3]. But this contradicts the fact that a positive character in \mathcal{S} cannot be isolated, [16, Theorem 2.1], hence $A \subset]-\infty, -1[$. By [12, Theorem 18(p.36)] we have

$$\lim_{n \rightarrow \infty} \frac{h(n+1)}{h(n)} \left| \frac{p_{n+1}(x)}{p_n(x)} \right| = C \lim_{n \rightarrow \infty} \left| \frac{p_{n+1}(x)}{p_n(x)} \right| = \left(|x| + (x^2 - 1)^{1/2} \right)^{-1} < 1,$$

whenever $x \in A$. This shows that α_x belongs to $\ell^1(\mathbb{N}_0)$. Hence, by Theorem 3.1 \mathbb{N}_0 is α_x -amenable with a unique α_x -mean.

(iii) and (iv) are shown in [8, Theorems 4.10–11]. □

Remark 4.2. (i) Theorem 4.1 reveals that the α -amenability of K in general depends on the asymptotic behavior of the Haar measure and α .

(ii) Observe that in Theorem 4.1 (iii) if $x \in (-1, 1)$ then the functionals m_{α_x} are distinct.

4.3. Conjecture: Let K be a α -amenable hypergroup. Then K has either a unique α -mean or the cardinality of the set of α -means is at most 2^{2^d} , where d is the smallest cardinality of a cover of K by compact sets.

References

- [1] A. Azimifard, α -Amenability of Banach algebras on commutative hypergroups. Ph.D. Thesis, Technical University of Munich, 2006.
- [2] W. R. Bloom and H. Heyer, Harmonic Analysis of Probability Measures on Hypergroups. De Gruyter, 1994.
- [3] F. Bonsall and J. Duncan, Complete Normed Algebras. Springer, Berlin, 1973.
- [4] T. S. Chihara, An Introduction to Orthogonal Polynomials. Gordon and Breach, 1978.
- [5] P. Civin and B. Yood, The second conjugate space of a Banach algebra as an algebra. Pac. J. Math. Vol. 11 (1961), 847–870.
- [6] N. Dunford and J. T. Schwartz, Linear Operators I, Wiley & Sons, 1988.

- [7] C. F. Dunkl, The measure algebra of a locally compact hypergroup. *Trans. Amer. Math. Soc.* 179 (1973), 331–348.
- [8] F. Filbir, R. Lasser, and R. Szwarc, Reiter’s condition P_1 and approximate identities for hypergroups. *Monat. Math.* 143 (2004), 189–203.
- [9] F. Filbir, R. Lasser and R. Szwarc, Hypergroups of compact type. *J. Comp. and App. Math.* (1) 178 (2005) 205–214.
- [10] R. I. Jewett, Spaces with an abstract convolution of measures, *Adv. in Math.* 18 (1975), 1–101.
- [11] G. R. A. Kamyabi, Topological center of dual Banach algebras associated to hypergroups. Ph.D. Thesis. University of Alberta, 1997.
- [12] P. G. Nevai, Orthogonal Polynomials. *Mem. Amer.Math.Soc.*, 1979.
- [13] A. L. T. Paterson, Amenability. *Amer. Math. Soc.*, 1988.
- [14] M. Skantharajah, Amenable hypergroups. *Illinois J. Math.* (1) 36 (1992), 15–46.
- [15] R. Spector, Aperçu de la théorie des hypergroupes. In *Anal. harmon. Groupes de Lie*, volume 497 of *Lecture Notes in Math.*, 643–673. Springer, Berlin, 1975.
- [16] M. Voit, Positive characters on commutative hypergroups and some applications. *Math. Z.* 198 (1988), 405–421.
- [17] M. Voit, Factorization of probability measures on symmetric hypergroups. *J. Aust. Math. Soc. Ser. A* (3) 50 (1991), 417–467.